UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 5th FEBRUARY 2004

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

1. **D** 4004 – 2004 = 2000, so 4002 – 2004 = 2000 – 2 = 1998.

2. **E** There must be 25 pupils who all have different birthdays. If the remaining 5 pupils all have the same birthday as one of these pupils, then 6 pupils will share the same birthday.

3. **E** 37 373 + 61 392 = 98 765 and 45 678 + 53 087 = 98 765.

4. **C** The angles marked $a°$, $c°$ and $e°$ may be considered to be the exterior angles of the triangle and therefore have a total of 360°. As $b° = a°$, $d° = c°$ and $f° = e°$ (all pairs of vertically opposite angles), $b + d + f = 360$. So $a + b + c + d + e + f = 720$.

5. **B** Let the numbers be x and y. Then $x + y = 2$; $x - y = 4$. Adding these equations gives $2x = 6$, so $x = 3$. Hence $y = 2 - 3 = -1$ so the numbers are 3 and -1.

6. **B** The sum is what we would write as 162 + 257 and this equals 419. However, in Niatirb it would be written 580.

7. **D** The shaded area is a trapezium of area $\frac{1}{2}(3 + 7) \times 5 = 25$. Line XD forms one side of a trapezium of area 12.5, since $\frac{1}{2}(1 + 4) \times 5 = 12.5$.

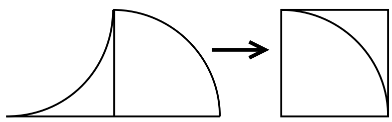
8. **D** The average number of rubber bands added each day was approximately $\dfrac{6\,000\,000}{5 \times 365} \approx \dfrac{6\,000\,000}{1800} \approx 3300$.

9. **E** The surface areas of the cuboids are: A 68; B 70; C 56; D 52; E 76.

10. **E** The four fractions total $\dfrac{5}{4}$, so their mean $= \dfrac{5}{4} \div 4 = \dfrac{5}{16}$.

11. **A** The 49 black squares will be in a 7×7 formation, so the board will measure 15×15 squares. Hence the number of white squares $= 225 - 49 = 176$.

12. **A** As the diagram shows, the figure may be cut into two parts which fit together to form a square measuring 8cm × 8cm.



13. **C** Points $(-3, -3)$, $(-2, -1)$, $(4, 11)$ and $(5, 13)$ all lie on the line whose equation is $y = 2x + 3$, but $(2, 5)$ does not lie on this line.

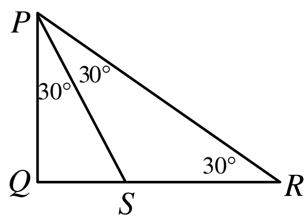
14. **B** We note first that $y = 5$ since that is the only non-zero digit which, when it is multiplied by 3, has itself as the units digit. So there is a carry of 1 into the tens column. We note also that $a = 1$ or $a = 2$ as "fly" < 1000 and therefore $3 \times$ "fly" < 3000. We now need $3 \times l + 1$ to end in either 1 or 2 and the only possibility is $l = 7$, giving $a = 2$ with a carry of 2 into the hundreds column. As $a = 2$, f must be at least 6. However, if $f = 6$ then $w = 0$ which is not allowed. Also the letters represent different digits, so $f \neq 7$ and we can also deduce that $f \neq 9$ since $f = 9$ would make $w = 9$.
Hence $f = 8$, making $w = 6$ and the letters represent $875 \times 3 = 2625$.

15. D As triangle A has two equal sides, it should have two equal angles, but its angles are 25°, 110° and 45° so it is impossible to make. In a triangle, the smallest angle lies opposite the shortest side which makes B impossible since 20° is the smallest angle, but 5 cm is not the shortest side. Triangle C is impossible as $4^2 + 7^2 \neq 8^2$, so it does not obey Pythagoras' Theorem. The longest side of a triangle must be shorter than the sum of the other two sides, but this is not the case in triangle E, so it cannot be made either. Triangle D, however, is obtained by cutting an equilateral triangle of side 6cm in half along an axis of symmetry and so can certainly be made.

16. C Note that the number at the end of the n th row is n^2, so 400 will lie at the end of the 20 th row. The row below will end in 21^2, i.e. 441, so the number directly below 400 will be 440.

17. B As P and Q both lie between 0 and 1, their product will be greater than 0 but smaller than P and smaller than Q. Of the options available, only B satisfies these conditions. (Furthermore, its position is correct since P is approximately equal to $\frac{1}{2}$, which means that the product of P and Q lies approximately half way between 0 and Q.)

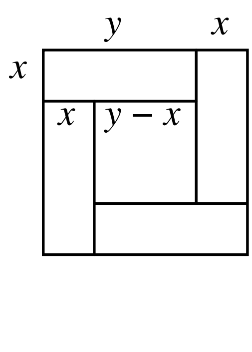
18. E Angle PRS = 30°, so triangle PRS is isosceles with $SP = SR$. Hence $\frac{QS}{SR} = \frac{QS}{SP} = \frac{1}{2}$ as PQS is half of an equilateral triangle.
[Alternatively, we can use the angle bisector theorem: $\frac{QS}{SR} = \frac{PQ}{PR} = \frac{1}{2}$ as PQR is also half of an equilateral triangle.]



19. D Consider the cuboid to be made up of 60 unit cubes. The front and side views show that top and bottom layers consist of the same number of cubes and from the top view we see that this number is 14. The front and side views indicate that only the 4 corner cubes remain in the middle layer, so the total number of cubes remaining is $2 \times 14 + 4 = 32$. The required fraction, therefore, is $\frac{32}{60} = \frac{8}{15}$.

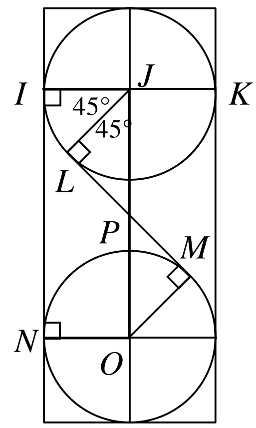
20. C Using the formula for the difference of two squares: $127^2 - 1 = 127^2 - 1^2 = (127 + 1)(127 - 1) = 128 \times 126 = 2^7 \times 2 \times 63 = 2^8 \times 3^2 \times 7$.

21. D Let the length of a short side of a rectangle be x and the length of a long side be y. Then the whole square has side of length $(y + x)$, whilst the small square has side of length $(y - x)$. As the area of the whole square is four times the area of the small square, the length of the side of the whole square is twice the length of the side of the small square. Therefore $y + x = 2(y - x)$ i.e. $y = 3x$ so $x : y = 1 : 3$.



22. A If you answer all questions correctly, you receive m marks for each of the N questions plus an extra 2 marks for the last $(N - q)$ questions. So the maximum possible score $= m \times N + 2 \times (N - q) = mN + 2N - 2q = (m + 2)N - 2q$.

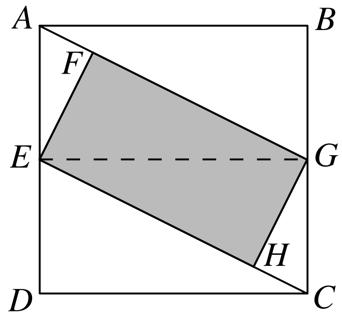
23. C Let the centres of the circles be J and O and let NI be the common tangent shown. Let P be the point of intersection of JO and LM. As arc KL is 5/8 of the circumference of the top circle, $\angle IJL = 45°$. Consider quadrilateral $IJON$: sides IJ and NO are radii, so they are both of unit length and they are both perpendicular to tangent NI. So $IJON$ is a rectangle. Hence $\angle IJO = 90°$ and $\angle LJP = 90° - 45° = 45°$. In triangle JLP, $\angle JPL = 180° - (90° + 45°) = 45°$. So triangle JLP is isosceles and $LP = LJ = 1$ unit.
By a similar argument, it may be shown that PM is also of length 1 unit, so $LM = LP + PM = 2$ units.



24. A We may deduce that $p > q$, so, as p and q are both positive, $\dfrac{p^2}{q^2} > \dfrac{\sqrt{p}}{\sqrt{q}} > 1$.

We may also deduce that $0 < \dfrac{q^2}{p^2} < \dfrac{q}{p} < \dfrac{\sqrt{q}}{\sqrt{p}} < 1$ and hence $\dfrac{q^2}{p^2}$ is the least of the five numbers.

25. B Triangles ABG, GFE and EFA are similar, so $AF : FE = EF : FG = GB : BA = 1 : 2$. Thus if $AF = a$, then $FE = 2a$, $FG = 4a$, and the shaded area is $8a^2$. By Pythagoras' Theorem, $AE = \sqrt{5}a$, so $AD = 2\sqrt{5}a$ and the area of the square is $20a^2$. Thus the required fraction is $8 / 20$, which is $2 / 5$.



[Alternatively, as we have shown that $FG = 4AF$, we can divide parallelogram $AGCE$ into 10 congruent triangles, 8 of which make up rectangle $EFGH$. So the area of the rectangle is 4/5 of the area of the parallelogram, which in turn is half the area of square $ABCD$.]